

CLASSIFICATION OF DOUBLE OCTIC CALABI–YAU THREEFOLDS

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ABSTRACT. In the present paper we propose a combinatorial approach to study the so called double octic Calabi–Yau threefolds. We use this description to give a complete classification of double octics with $h^{1,2} \leq 1$ and to derive their geometric properties (Kummer surface fibrations, automorphisms, special elements in families).

INTRODUCTION

Let $\pi : X \longrightarrow \mathbb{P}^3$ be a double covering of the projective space branched along an arrangement of eight planes. If the arrangement satisfies mild conditions (no six planes intersect, no four contain a line) then there exists a resolution of singularities of X which is a projective Calabi–Yau threefold called a double octic ([8]). Double octic Calabi–Yau are very suitable for explicit computations, their invariants (topological Euler characteristic, Hodge numbers) can be easily computed ([5]). On the other hand this class is rich enough to provide examples of several interesting phenomena.

There exist double octic Calabi–Yau threefolds in characteristic 3 non-liftable to characteristic zero ([6]). Computation of Picard–Fuchs operators of one-parameter families of double octic Calabi–Yau exhibited examples with particular properties: an example without a point of Maximal Unipotent Monodromy or an example with three points of Maximal Unipotent Monodromy and different instanton numbers ([7]). Double octic Calabi–Yau threefolds have elliptic curve and K3 surface fibrations, K3 fibrations (K3 fibrations of rigid double octics were studied in [1]).

Double octic Calabi–Yau threefolds are closely related to desingularized fiber products of rational elliptic surfaces ([20]), a double octic can be considered as a fiber-wise Kummer construction. This relation and its application to modularity of certain Calabi–Yau threefolds (also non-rigid cf. [10]) and existence of correspondences was studied in [14].

Double octic Calabi–Yau threefolds were studied by C. Meyer ([17]), he carried out a systematic study of huge number of examples with integer coefficients in particular he gave 11 examples of rigid double octics and 63 examples of one-parameter families. Our main task is to make a complete classification of octic arrangements, contrary to Meyer we did not study explicit examples. Instead we used combinatorial data called “incidence table” which are independent of the field considered. Conversely, from an incidence table we can recover equation of an octic arrangement and verify if two families with identical incidence

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tables are projectively equivalent. From the incidence table we can read singularities of the arrangement, presentations as a fiber-wise Kummer fibration, permutations of planes that preserves the incidences and projective transformations of projective space that preserve the octic arrangement.

We use this combinatorial approach to produce a complete list of examples with $h^{1,2} \leq 1$ and describe their geometry.

1. DOUBLE OCTICS

Let $D = D_1 \cup \dots \cup D_8$ be a sum of eight planes in \mathbb{P}^3 , we shall call D an *octic arrangement* iff

- the intersection $D_{i_1} \cap \dots \cap D_{i_6}$ of any six is empty ($1 \leq i_1 \leq i_2 \leq \dots \leq i_6 \leq 8$),
- the intersection $D_{i_1} \cap D_{i_2} \cap D_{i_3} \cap D_{i_4}$ of any four do not contain a line i.e. it is empty set or one point ($1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq 8$).

The surface D is singular at the intersection points of components, there are five type of singularities of octic arrangements satisfying the above two restrictions

- l_2 double line,
- l_3 triple line,
- p_3 triple point (not on a triple line),
- p_4^0 fourfold point not lying on a triple line,
- p_4^1 fourfold point lying on a one triple line,
- p_5^0 fivefold point not lying on a triple line,
- p_5^1 fivefold point lying on one triple line,
- p_5^2 fivefold point lying on a one triple line.

We denote the numbers of multiple lines and points by $l_2, l_3, p_3, p_4^0, p_4^1, p_5^0, p_5^1, p_5^2$. These numbers are related by the following two relations [8, Lem. 3.4]

$$\begin{aligned} p_3 + 4p_4^0 + 3p_4^1 + 10p_5^0 + 9p_5^1 + 8p_5^2 + l_3 &= 56 \\ p_4^1 + 2p_5^1 + 4p_5^2 &= 5l_3. \end{aligned}$$

The double cover of \mathbb{P}^3 branched along D admits a resolution of singularities X that is a smooth Calabi–Yau threefold, the topological Euler characteristic of X is given by ([8, Thm. 3.5])

$$e(X) = 40 + 4p_4^0 + 3p_4^1 + 16p_5^0 + 18p_5^1 + 20p_5^2 + l_3.$$

The Kuranishi space (universal deformation) of a double octic X is given by the space of equisingular deformations (i.e. family of octic arrangements preserving the types of singularities) modulo trivial deformations (induced by projective automorphisms of \mathbb{P}^3). Dimension of the Kuranishi space equals the Hodge number $h^{1,2}(X)$ and can be computed with computer algebra system via equisingular ideal (cf. [5]).

C. Meyer in [17] carried out an extensive computer search for double octic Calabi–Yau threefolds. His method was to study arrangements with small integer coefficients, compute for them numbers of singularities of various types ($l_2, l_3, p_3, p_4^0, p_4^1, p_5^0, p_5^1, p_5^2$) and the Hodge numbers $h^{1,1}, h^{1,2}$. As those numerical invariants do not classify arrangements he also classified the types of all subarrangements of six planes.

Among 450 types of octic arrangements listed in [17, App. A] there are 11 producing rigid (admitting no deformations, equivalently with the Hodge number $h^{1,2} = 0$) Calabi–Yau threefolds and 63 with $h^{1,2} = 1$. For the 74 arrangement Meyer gave sample equations of the eight planes, we shall use equations and the numbering of the arrangements from [17] (we only corrected equation of Arr. No. 35 and change the parametrizations of Arr. No. 275 and 276).

2. INCIDENCE TABLE

Classification of double octic Calabi–Yau threefolds is a delicate matter, Meyer used two measures to classify octic arrangements: the numerical invariants (number of singular points of various types and the Hodge numbers) and the ordered list of types of subarrangements of six planes. However two completely different octic arrangements (f.i. rigid arrangements no. 32 and 69 in the Meyer list [17]) can give birational Calabi–Yau threefolds. On the other hand the numerical invariants do not determine the double octic. To avoid this difficulties we shall introduce the following definitions

Definition 2.1. Arrangements of eight planes $D_1 = P_1^1 \cup \dots \cup P_8^1$ and $D_2 = P_1^2 \cup \dots \cup P_8^2$ are called combinatorially equivalent iff there is a permutation $\sigma \in S_8$ such that for each indices $1 \leq i_1 < i_2 < \dots < i_k \leq 8$ the intersections

$$P_{i_1}^1 \cap \dots \cap P_{i_k}^1 \quad \text{and} \quad P_{\sigma(i_1)}^2 \cap \dots \cap P_{\sigma(i_k)}^2$$

have the same dimension and (provided non-empty) give the same singularity type in D_1 and D_2 .

Definition 2.2. The *incidence table* of an octic arrangement $D = P_1 \cup \dots \cup P_8$ is the sorted list of all quadruples $1 \leq i_1 < i_2 < i_3 < i_4 \leq 8$ such that the planes P_{i_1}, \dots, P_{i_4} intersect.

The *minimal incidence table* of an octic arrangement is the minimum of incidences tables over all permutations of the planes.

The minimal incidence table uniquely determines the combinatorial equivalence type of an arrangement.

Proposition 2.1. *Arrangements of eight planes $D_1 = P_1^1 \cup \dots \cup P_8^1$ and $D_2 = P_1^2 \cup \dots \cup P_8^2$ are combinatorially equivalent iff they have equal minimal incidence tables.*

Proof. Obviously equivalent arrangements have equal minimal incidence tables, to prove the opposite implication we can assume, that (after reordering one of them) the arrangement in question have equal incidence tables.

Observe first, that a set of planes in \mathbb{P}^3 intersect iff any four of them intersect. Now, let $1 \leq i < j < k \leq 8$ be any triple of indices, that the planes P_i^1, P_j^1, P_k^1 intersect along a line iff for each $l \in \{1, \dots, 8\} \setminus \{i, j, k\}$ the planes $P_i^1, P_j^1, P_k^1, P_l^1$ intersects. As the same holds true for planes P_i^2 , we get triple lines, fourfold and fivefold points in both arrangements given by the corresponding planes. As the type of a singularity is determined by the number of planes and triple lines through a point, both arrangements are combinatorially equivalent. \square

An octic arrangement can be described by a 8×4 matrix of coefficients of linear forms defining the planes, four planes of the arrangements intersect iff the corresponding 4×4 minor is zero. Consequently the incidence table can be computed from the coefficient matrix by finding vanishing maximal minors.

For instance for the Arr. 1. given by

$$xyzt(x+y)(y+z)(z+t)(t+x) = 0$$

the matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A & 0 & 0 & B \end{bmatrix}$$

The $70 = \binom{8}{4}$ degree 4 minors (in a lexicographic order) equal $1, 0, 0, -1, B, 0, -1, 1, 0, 0, 0, 0, 1, -B, B, -1, 1, 0, 0, 0, -1, B, -1, B, 0, -1, 1, 0, 1, 0, 0, -1, B, -B, B, 1, 0, 0, -A, 0, 1, -B, 0, 0, -A, 1, -1, 0, 0, -A, A, 1, -B, B, -A, -1, 0, A, 0, A, 0, -1, B, -A, A, 1, -A, A, -A, A - B$. Incidence table is the following list of 24 quadruples of indices $1235, 1236, 1245, 1248, 1256, 1257, 1258, 1347, 1348, 1356, 1378, 1458, 1468, 1478, 2346, 2347, 2356, 2367, 2368, 2458, 2467, 3457, 3467, 3478$.

Although the (sorted) incidence table is independent of the coordinates in \mathbb{P}^3 , but it depends on the order of eight planes. We ran through all the permutations of planes and found that the minimal incidence tables is

$1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1468, 1568, 2345, 2367, 2467, 2468, 2478, 2567, 2678, 3468, 4568, 4678$.

The *minimizing permutation* is given by the following product of disjoint cycles $(126834)(57)$ of eight letters.

Observe, that we can try to revert the above considerations. For an incidence table write down a matrix with generic coefficients and compute the appropriate minors resulting in a system of equations (in 16 variables) describing all octic arrangements with the given incidence matrix. More precisely we get an arrangement with incidence table containing the table we started with, using computer algebra system we check if (and which) remaining minors belongs to the ideal generated by the assigned ones.

The disadvantage of this direct approach is that we get a very complicated system of equations that would be very difficult to handle. To simplify the computations we can identify a rigid subarrangement of five or six planes. Many of the considered arrangements contain a rigid subarrangement of six faces of a cube (which reduces the number of parameters to 8 and the degrees of the equations to 2). In remaining cases we identify a generic subarrangement of five planes (12 parameters, degree of equations at most 3).

3. SPECIAL ELEMENTS

In every non-trivial one parameter family of Calabi–Yau threefolds there are special elements which are not smooth. In the case of a double octic Calabi–Yau threefolds they corresponds to arrangements with altered types (or numbers) of singularities. Equivalently they are the arrangements with bigger incidence table, consequently they are described by vanishing of all the non-zero minors of the coefficients matrix.

We can encounter two different situation

- special arrangement does not satisfy the definition of an octic arrangement (i.e. it does not obey the restrictions we put on the intersections — two planes equals, four planes intersect in a line or six planes intersect), then the special element does not admit a Calabi–Yau resolution.
- special arrangement is an octic arrangement of different type (non-equivalent). The blow-ups we perform to resolve a generic element of the family are not enough to resolve them, special element of the family is again singular but this time it admits a Calabi–Yau resolution (Calabi–Yau variety).

In [5] a conifold expansion was used to compute the Picard–Fuchs operator for families in which four planes in general position degenerate to four intersecting planes without any further degenerations – geometrically it corresponds to a shrinking tetrahedron. This kind of degenerations occur when the incidence table increases by a one quadruple. There can be more than one shrinking tetrahedron, in this case there are more than one new incidence, but any two of them have at most two common incidences.

4. AUTOMORPHISMS, ELLIPTIC FIBRATIONS

Arithmetic properties of Calabi–Yau threefolds (f.i. in the context of Modularity Conjecture – see [17] for details) depend on special geometric properties, we shall study Kummer surface fibrations and actions of finite groups.

If the eight planes split into two quadruples of intersecting planes (two opposite fourfold points), then each quadruple produces an elliptic surface, the double octic Calabi–Yau has a double cover by a fiber product of rational elliptic surfaces. The double octic is a Kummer surface fibration for the two elliptic fibrations. Consider an arrangement $D = P_1 \cup \dots \cup P_8$ of eight planes given by equations

$$P_i = \{f_i = 0\}, i = 1, \dots, 8$$

and assume that among the eight planes there are two disjoint quadruples intersecting in a point each. After renumbering the equations and changing coordinates we can assume that the planes P_1, \dots, P_4 intersect at the point $A = (0, 1, 0, 0)$ whereas P_5, \dots, P_8 intersect at $B = (1, 0, 0, 0)$. Equivalently, equations f_1, \dots, f_4 depend only on x, z, t , f_5, \dots, f_8 depend only on y, z, t . Equations f_1, \dots, f_4 (resp. f_5, \dots, f_8) define four lines in the projective plane $\mathbb{P}_{x,z,t}^2$ (resp. $\mathbb{P}_{y,z,t}^2$), that we can identify with projection from A and B respectively.

Let S and S' be the double coverings of \mathbb{P}^2 branched along the corresponding sums of four lines. Then in appropriate affine coordinates (f.i., $t = 1$) they can be written as follows:

$$S = \{(x, z, u) \in \mathbb{C}^3 : u^2 = f_1(x, z, 1) \cdot \dots \cdot f_4(x, z, 1)\}$$

$$S' = \{(y, z, v) \in \mathbb{C}^3 : v^2 = f_5(y, z, 1) \cdot \dots \cdot f_8(y, z, 1)\}$$

This exhibits (birationally) both surfaces as elliptic fibrations. Moreover, the map

$$((x, y, u), (y, t, v)) \mapsto (x, y, t, uv)$$

is a rational, generically $2 : 1$ map from their fiber product to the double covering of \mathbb{P}^3 branched along the octic surface D .

Singular fibers of the elliptic surfaces S (S') correspond to the projection of the six lines $l_{ij} = P_i \cap P_j$ with $1 \leq i < j \leq 4$ ($m_{ij} = P_{i+4} \cap P_{j+4}$) from the line AB , we call the lines l_{ij} and l_{jk} (resp. m_{ij} and m_{jk}) *conjugate* if $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The type of a singular fiber is determined by the number of lines that project to the same point

- I_2 in the case of one line,
- I_4 – two lines,
- I_0^* – three lines,
- I_2^* – four lines.

Consequently, the fibers of type I_0^* and I_2^* can be recognized from the incidence table. A fiber of the type I_0 can occur in two different situation, three of the planes P_1, \dots, P_4 intersect along a (triple) line (so the three lines actually coincide), one of the planes P_1, \dots, P_4 pass through the (fivefold) point B . A fiber of the type I_2^* corresponds to three planes (out of P_1, \dots, P_4) intersecting along a line, and one of them passing through the point B . Consequently we can get surfaces with the following sequences of singular fibers

$$\begin{array}{lll} S_1 : & (I_0^*, I_0^*), & S_2 : (I_4, I_4, I_2, I_2), & S_3 : (I_0^*, I_2, I_2, I_2), \\ S_4 : & (I_2^*, I_2, I_2), & S_5 : (I_4, I_2, I_2, I_2, I_2), & S_6 : (I_2, I_2, I_2, I_2, I_2, I_2). \end{array}$$

In table 4.1 we give examples of explicit equations for the branch locus of corresponding double quartic elliptic fibrations and types and coordinates of the singular fibers. Fibration S_1 is special as it has only two singular fibers. For the remaining cases we specialize three of singular fibers to ∞ , 0 and 1 . Surfaces S_2 and S_4 are uniquely fixed by the position of singular fibers. Singular fibers of S_2 form a harmonic quadrilateral, there is an isogeny of this elliptic surface that exchanges I_2 fibers with I_4 fibers (yielding some correspondences between certain double octics). For any fixed position of singular fibers there exists two non-equivalent fibrations of type S_3 corresponding to the two described above geometric realization. The following transformation

$$(u, x, z, t) \mapsto (\lambda ztu, \lambda zt, xz, xt)$$

gives a birational transformation of the two elliptic surface given in the table, and consequently birational transformations of some double octics coming from non-equivalent octic arrangements. A surface of type S_5 has one I_4 fiber and two pairs of conjugate I_2 singularities, for any symmetric position of singular fibers there exists a unique surface of type S_5 .

S_1	I_0^*	I_0^*	$xz(x+t)(x+2t)$				
	∞	0					
S_2	I_4	I_4	I_2	I_2	$x(x+t)(x+z)(x+z+t)$		
	∞	0	1	-1			
S_3	I_0^*	I_2	I_2	I_2	$x(x+t)(x+\lambda t)(x+z)$ $t(x+\lambda z)(x+z)(x+\lambda t)$		
	∞	0	1	λ			
S_4	I_2^*	I_2	I_2	$xt(x+z)(x+t)$			
	∞	0	1				
S_5	I_4	I_2	I_2	I_2	I_2	$x(x+t)(x+z-\lambda t)(x+z)$	
	∞	0	1	λ	$\lambda+1$		
S_6	I_2	I_2	I_2	I_2	I_2	I_2	$x(x+t)(x+z)(x+\frac{1}{\mu-\lambda}(z-\lambda t))$ $x(x+t)(\lambda x+z)((\mu-1)x+z-t)$
	∞	0	1	λ	μ	$\frac{\lambda}{\lambda-\mu+1}$	

TABLE 4.1. Elliptic fibrations

Surface of type S_6 has three pairs of conjugate I_2 singularities symmetric with respect to some involutive automorphism of the projective line. For any symmetric position of six singular fibers there are two surfaces of type S_6 , geometrically we can say that they are keeping track of the order of the conjugate pairs, we can change order of any two of them. If we specialize three pairwise non-conjugate singular fibers at ∞ , 0 and 1 we have two options: corresponding three lines l_{ij} are – up to permutation – l_{12}, l_{13}, l_{23} or $l_{12}, l_{1,3}, l_{1,4}$ leading to the two equations in the table. These two surfaces are birational and the birational map can be given by

$$(u, x, z, t) \longmapsto \left(u(t-z)(\lambda t-z)\sqrt{\frac{\lambda-\mu}{\lambda(\mu-1)}}, (t-z)x, (\lambda x-x+\lambda t-z)z, (\lambda x-x+\lambda t-z)t \right)$$

Fiber product of rational elliptic surfaces (hence also the Kummer fibration) is determined by the matching of singular fibers, singular fibers given by the lines l_{ij} and m_{kl} are mapped to the same point in \mathbb{P}^1 iff the lines l_{ij} and m_{kl} intersects i.e. the corresponding four planes intersect in a point (the appropriate quadruple belongs to the incidence table).

For every considered octic arrangement we determine permutations in S_8 which leave the incidence table invariant, using MAGMA (and Gap's SmallGroups Library) we classify the group of symmetries and small system of generators. Using this data one can determine which symmetries lift to projective transformation of \mathbb{P}^3 and the double octic.

5. ALGORITHM

Any incidence table determines two additional lists: the list of triplets of planes intersecting along a (triple) line and the list of quintuples of planes intersecting at a (fivefold) point. A triplet belongs to the first list iff all quadruples containing it belongs to the incidence table, similarly a quintuple belongs to the second list if all quadruples contained in it belongs to the incidence table.

If two quadruples belonging to the incidence table have three common interface then their intersection belongs to the list of triplets or their sum belongs to the list of quintuples (or both). Finally neither two triplets can have two common entries nor two quintuples – four common entries. Finally the lists of triplets and quintuples have at most 4 elements each (an arrangement has at most four triple lines and at most four fivefold points).

Our strategy is to operate on a list which entries consist of three list: list of triplets, quadruples and quintuples. We start with the list with one entry: the list of quadruples contains one element 1234 (the smallest possible), the lists of triplets and quintuples are empty. Then we repeat the following steps on each entry of the list

- introduce new element in the list by adding a single quadruple (that is not yet on a list of quadruples),
- if two quadruples in the list have three common entries we replace corresponding entry with two new ones by adding the intersection to the triplets list or the sum to the quintuples list
- if two triplets have two or quintuples – four common entries, we remove the corresponding element from the list
- for each triplets we add all quadruples containing it and for each quintuple – all quadruples contained in it
- we apply all permutations of eight digits to the triplets, quadruples and quintuples list and replace the entry with minimal list of quadruples.

We repeated these steps until the list stabilized. Then we applied the inverse procedure to the entries which were obtained after first seven steps, which is a necessary condition for $h^{1,2} \leq 1$. For each such entry we determined a quintuples of planes in general position in the arrangement (i.e. a quintuple of eight digits which does not contain a quadruple from the incidence table). Then we assumed that these five planes have equations $x, y, z, t, x+y+z+t$, whereas the remaining three planes have equations $A_i x + B_i y + C_i x + D_i t$ and computed the ideal generated by the minors of entries of the coefficients matrix corresponding to the entries in the incidence table. Associated primes of this ideal that do not impose any extra elements in the incidence table give us the requested octic arrangements.

Finally we computed the minimal incidence tables for the examples from [17] and recognize them in the produced list.

6. RESULTS

In this section we collect the results of Magma computations, for the sake of completeness we also include the data of the eleven rigid arrangements listed in [17] and three more that we shall denote as A, B and C. Observe that these three examples are not defined over \mathbb{Q} so they cannot be found in [17], however they appeared in [5, 7]. Our computations produced

only one new example which we denote by D , a family defined over $\mathbb{Q}[\sqrt{-3}]$, again this example could not appear in the list of Meyer. Finally, there is only one rigid double octic Calabi–Yau threefold in positive characteristic, that cannot be lifted to characteristic zero, it is the non-liftable example in characteristic 3 in [6].

Proposition 6.1. *The lists of double octic Calabi–Yau threefolds with $h^{1,2} \leq 1$ in [17] defined over \mathbb{Q} is complete, there exist four examples that cannot be realized over \mathbb{Q} , they have the following Hodge numbers:*

$$A : h^{1,1} = 46, h^{1,2} = 0$$

$$B : h^{1,1} = 38, h^{1,2} = 0$$

$$C : h^{1,1} = 38, h^{1,2} = 0$$

$$D : h^{1,1} = 36, h^{1,2} = 1$$

Most of the data are self explaining, we do not include the (original) incidence tables as they can be easily computed. We also do not include the full symmetry groups, instead we give an isomorphism type and a small set of generators. In fact we get 16 isomorphism types, most of them can be represented as direct products of cyclic, dihedral and symmetric groups, three biggest and most complicated we denoted by $g_{32,43}$, $g_{64,138}$ and $g_{192,955}$ (notation follows the SmallGroups library from GAP). These groups can be described as:

$g_{32,43}$: is the holomorph of the cyclic group C_8 , i.e. $\text{Aut}(C_8) \rtimes C_8$;

$g_{64,138}$: is the unitriangular matrix group $UT(4, 2)$ of degree four over the field of two elements.

$g_{192,955}$: is a semidirect product $C_2^{\oplus 4} \rtimes D_6$ and is isomorphic to the automorphism group $\text{Aut}(C_2 \times Q_8)$

Some arrangements have several pairs of opposite fourfold points and so also corresponding fiber products of rational elliptic surfaces, we list only one example for each isomorphisms class. In case of surfaces of types S_5 and S_6 we marked pairs of conjugate singular fibers.

6.1. Rigid arrangements. Arr. No. 1: $xyzt(x+y)(y+z)(z+t)(x+t)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1468, 1568, 2345, 2367, 2467, 2468, 2478, 2567, 2678, 3468, 3578, 4568, 4678

Minimizing permutation: (126834)(57)

Symmetries: D_4 , $\langle (1432)(5876), (12)(34)(68) \rangle$

Singular points:

$$p_4^0 : 5678$$

$$p_4^1 : 1275, 1468, 2386, 3457$$

$$p_5^2 : 12356, 12458, 13478, 23467$$

$$l_3 : 125, 148, 236, 347$$

Elliptic fibrations:

∞	0	1
I_2^*	I_2	I_2
I_2	I_2^*	I_2

Arr. No. 3: $xyzt(x+y)(y+z)(y-t)(x-y-z+t)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1467, 1567, 1678, 2345, 2367, 2468, 2578, 3478, 4567

Minimizing permutation: (172)(3564)

Symmetries: S_3 , $\langle(176)(354), (17)(45)\rangle$

Singular points:

$$\begin{aligned} p_4^0 &: 1378, 1468, 5678 \\ p_4^1 &: 1285, 2386, 2487 \\ p_5^2 &: 12356, 12457, 23467 \\ l_3 &: 125, 236, 247 \end{aligned}$$

Elliptic fibrations:	∞	0	1	2
	I_2^*	—	I_2	I_2
	I_4	I_2	I_4	I_2

Arr. No. 19: $xyzt(x+y)(y+z)(x-z-t)(x+y+z-t)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 2345, 2367, 2468, 2578, 3478, 3568,

Minimizing permutation: (1548632)

Symmetries: $C_2 \oplus C_2$, $\langle(15)(78), (15)(36)\rangle$

Singular points:

$$\begin{aligned} p_4^0 &: 1347, 1468, 3458, 4567 \\ p_4^1 &: 1245, 1275, 1285, 2346 \\ p_5^1 &: 23768 \\ p_5^2 &: 12356 \\ l_3 &: 125, 236 \end{aligned}$$

Elliptic fibrations:	∞	0	1	1/2
	I_2^*	I_2	I_2	—
	I_2	I_4	I_4	I_2

Arr. No. 32: $xyzt(x+y)(y+z)(x-y-z-t)(x+y-z+t)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1456, 1457, 1458, 1678, 2345, 2467, 2568, 3468, 3578,

Minimizing permutation: (15478632)

Symmetries: $C_2 \oplus C_2$, $\langle(13)(56)(78), (16)(35)\rangle$

Singular points:

$$\begin{aligned} p_4^0 &: 1378, 1467, 2478, 3458, 5678 \\ p_4^1 &: 1245, 1275, 1285, 2346, 2376, 2386 \\ p_5^2 &: 12356 \\ l_3 &: 125, 236 \end{aligned}$$

Elliptic fibrations:	∞	0	1	−1
	I_4	I_4	I_2	I_2
	I_0^*	I_2	I_2	I_2

Arr. No. 69: $xyzt(x+y)(x-y+z)(x-y-t)(x+y-z-t)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1468, 1578, 2345, 2367, 2478, 3568, 4567

Minimizing permutation: (13745)

Symmetries: $C_2 \oplus C_2$, $\langle (12)(34)(67), (12)(36)(47) \rangle$

Singular points:

p_4^0 : 1468, 2378, 3458, 3467, 5678

p_4^1 : 1285

p_5^1 : 12356, 12457

l_3 : 125

Elliptic fibrations:

∞	0	1	-1
I_4	I_4	I_2	I_2
I_0^*	I_2	I_2	I_2

Arr. No. 93: $xyzt(x+y)(x-y+z)(y-z-t)(x+z-t)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2567, 3678, 4578

Minimizing permutation: (12)(35)(46)(78)

Symmetries: $C_2 \oplus C_2$, $\langle (12)(36), (12)(78) \rangle$

Singular points:

p_4^0 : 1348, 1467, 2347, 2468, 3678, 4578

p_4^1 : 1245, 1275, 1285

p_5^1 : 12356

l_3 : 125

Elliptic fibrations:

∞	0	1	1/2
I_2	I_4	I_4	I_2
I_0^*	I_2	I_2	I_2

Arr. No. 238: $xyzt(x+y+z-t)(x+y-z+t)(x-y+z+t)(-x+y+z+t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2367, 2457, 2468, 3456, 3478, 5678

Minimizing permutation: (172)(346)

Symmetries: $G_{192,955}$, $\langle (23)(5687), (1835)(2746) \rangle$

Singular points:

p_4^0 : 1256, 1278, 1357, 1368, 1458, 1467, 2358, 2367, 2457, 2468, 3456, 3478

Elliptic fibrations:

∞	0	1	-1
I_2	I_2	I_4	I_4
I_2	I_2	I_4	I_4

Arr. No. 239: $xyzt(x+y+z)(x+y+t)(x+z+t)(y+z+t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2358, 2367, 2457, 3456

Minimizing permutation: (13)(45)(68)

Symmetries: S_4 , $\langle (23)(67), (1234)(5876) \rangle$

Singular points:

p_4^0 : 1235, 1246, 1278, 1347, 1368, 1458, 2348, 2367, 2457, 3456

Elliptic fibrations:

∞	0	$-1/2$	-1	-2
I_2	I_4	I_2	I_4	$-$
I_4	I_2	$-$	I_4	I_2

Arr. No. 240: $xyzt(x+y+z)(x+y-z+t)(x-y+z+t)(x-y-z-t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2367, 2457, 3456, 4678

Minimizing permutation: (14)(27)(36)(58)

Symmetries: S_3 , $\langle(12)(78), (123)(687)\rangle$

Singular points:

p_4^0 : 1235, 1278, 1368, 1458, 1467, 2367, 2457, 2468, 3456, 3478

Elliptic fibrations:

∞	0	1	-1	-3
I_2	I_2	I_4	I_4	$-$
I_2	I_2	I_2	I_4	I_2

Arr. No. 241: $xyzt(x+y+z+t)(x+y-z-t)(y-z+t)(x+z-t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 2467, 3456, 3678, 4578

Minimizing permutation: (13267584)

Symmetries: $G_{32,43}$, $\langle(34)(56), (17)(36)(45), (14857326), (28)(36)(45)\rangle$

Singular points:

p_4^0 : 1256, 1278, 1348, 1357, 1467, 2347, 2368, 2458, 3456, 5678

Elliptic fibrations:

∞	0	1	-1
I_4	I_4	I_2	I_2
I_2	I_2	I_4	I_4

Arr. No. 245: $xyzt(x+y+z)(y+z+t)(x-y-t)(x-y+z+t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2358, 2367, 4567

Minimizing permutation: (1362)(458)

Symmetries: C_2 , $\langle(16)(37)\rangle$

Singular points:

p_4^0 : 1235, 1247, 1268, 1367, 1456, 2346, 2458, 2567, 3478

Elliptic fibrations:

∞	0	1	-1	-3
I_4	I_4	I_2	I_2	$-$
I_2	I_2	I_2	I_4	I_2

Arr. No. A: $xyzt(x+y)(x+y+z-t) \times$

$$\times ((\sqrt{-3}-1)x - 2y + (\sqrt{-3}-1)z)(2y + (-\sqrt{-3}+1)z - 2t)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2578, 3478, 3567

Minimizing permutation: (13468752)

Symmetries: S_3 , $\langle (12)(37)(48), (25)(37)(46) \rangle$

Singular points:

$$p_4^0 : 1368, 1478, 2348, 2467, 3456, 5678$$

$$p_4^1 : 1245, 1265, 1285$$

$$p_5^1 : 12357$$

$$l_3 : 125$$

Arr. No. B: $xyzt(x+y+z)(x+z-t)((\sqrt{-3}-1)x + (\sqrt{-3}+1)y - 2z + 2t)$
 $\times ((\sqrt{-3}-1)x + (\sqrt{-3}-1)y - 2z + (\sqrt{-3}+1)t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2358, 2457, 3456, 4678

Minimizing permutation: (13528476)

Symmetries: D_4 , $\langle (28)(36), (16)(27)(35)(48) \rangle$

Singular points:

$$p_4^0 : 1235, 1267, 1346, 1378, 1568, 2456, 2478, 3458, 3567$$

Elliptic fibrations:

0	1	∞	$\frac{1}{2} - \frac{\sqrt{-3}}{2}$	$-\frac{1}{2} - \frac{\sqrt{-3}}{2}$
I_2	I_2	I_4	I_2	I_2
I_4	I_2	I_2	I_2	I_2

Arr. No. C: $xyzt(x+y+z)((\sqrt{5}-1)y - 2z + 2t)(2x + 2y + (\sqrt{5}-1)t)$
 $\times ((-\sqrt{5}+3)x + 2y + (-\sqrt{5}+1)z + (\sqrt{5}-1)t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2358, 2457, 3467, 4568

Minimizing permutation: (125387)(46)

Symmetries: C_2 , $\langle (17)(25)(38)(46) \rangle$

Singular points:

$$p_4^0 : 1235, 1247, 1268, 1378, 1567, 2346, 2578, 3457, 4568$$

Elliptic fibrations:

0	1	∞	$-\frac{1}{2} - \frac{\sqrt{5}}{2}$	$\frac{1}{2} - \frac{\sqrt{5}}{2}$
I_2	I_2	I_2	I_2	I_4
I_2	I_2	I_4	I_2	I_2

6.2. Non-rigid arrangements.

Arr. No. 2: $xyzt(x+y)(y+z)(z+t)(Ax+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1468, 1568, 2345, 2367, 2467, 2468, 2478, 2567, 2678, 3468, 4568, 4678

Minimilizing permutation: (126834)(57)

Symmetries: D_4 , $\langle(1432)(5876), (14)(23)(57)\rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: Arr. 1,

Singular points:

$$p_4^1 : 1275, 1468, 2386, 3457$$

$$p_5^2 : 12356, 12458, 13478, 23467$$

$$l_3 : 125, 148, 236, 347$$

Elliptic fibrations:	∞	0	1	$\frac{A}{B}$
	I_2^*	I_2	I_2	—
	I_2	I_2^*	—	I_2

Arr. No. 4: $xyzt(x+y)(y+z)(Ax+By+Bz-At)(Ax+Ay+Bz-At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1467, 1567, 1678, 2345, 2367, 2468, 3578, 4567

Minimilizing permutation: (136752)(48)

Symmetries: D_6 , $\langle(137568), (16)(35)\rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: non-CY,

Singular points:

$$p_4^0 : 1467, 3458$$

$$p_4^1 : 1245, 2346, 2478$$

$$p_5^2 : 12356, 12758, 23768$$

$$l_3 : 125, 236, 278$$

Elliptic fibrations:	∞	0	1	$\frac{B}{A}$	$\frac{A-B}{A}$
	I_2^*	I_2	I_2	—	—
	I_4	I_2	I_2	I_2	I_2

Arr. No. 5: $xyzt(x+y)(y+z)(x+y+z-t)(Ax+By+Az-At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1467, 1567, 1678, 2345, 2367, 2468, 2578, 4567

Minimilizing permutation: (172)(348)(56)

Symmetries: $C_2 \oplus C_2$, $\langle(16)(35), (13)(56)\rangle$

Special values: ∞ : Arr. 3, 0: non-CY, 1: non-CY, 1/2: Arr. 3,

Singular points:

$$p_4^0 : 1467, 3457$$

$$p_4^1 : 1245, 2346, 2478$$

$$p_5^2 : 12356, 12758, 23768$$

$$l_3 : 125, 236, 278$$

Elliptic fibrations:	∞	0	1	$\frac{B}{A}$	$-\frac{A-B}{A}$
	I_4	I_2	I_2	I_2	I_2
	I_2^*	I_2	I_2	—	—
and	∞	-2	-1	0	$-\frac{B}{A}$
	I_4	I_2	I_4	I_2	—
	I_2^*	—	I_2	—	I_2

Arr. No. 8: $xyzt(x+y)(y+z)(z-t)(Ax-By-Bz+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1468, 1568, 2345, 2367, 2467, 2567, 2678, 3478, 4568

Minimilizing permutation: (147632)

Special values: ∞ : non-CY, 0: non-CY, -1: Arr. 1,

Singular points:

p_4^0 : 1468

p_4^1 : 1245, 1347, 2386, 3457, 3487

p_5^1 : 12758

p_5^2 : 12356, 23467

l_3 : 125, 236, 347

Elliptic fibrations:	∞	-1	0	$-\frac{B}{A+B}$
	I_0^*	I_2	I_2	I_2
	I_2	I_2	I_2^*	—
and	∞	0	1	$\frac{A+B}{B}$
	I_2^*	—	I_2	I_2
	I_2	I_2^*	I_2	—

Arr. No. 10: $xyzt(x+y)(y+z)(z-t)(Ax-By-Bz-At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 2345, 2367, 2467, 2567, 2678, 3468, 3578

Minimilizing permutation: (174563)

Symmetries: $C_2 \oplus C_2$, $\langle (15)(47), (17)(23)(45) \rangle$

Special values: ∞ : Arr. 1, 0: non-CY, -1: Arr. 1,

Singular points:

p_4^0 : 1468, 5678

p_4^1 : 1245, 1275, 1285, 1347, 2386, 3457, 3487

p_5^2 : 12356, 23467

l_3 : 125, 236, 347

Elliptic fibrations:	∞	0	-1	$-\frac{B}{A+B}$
	I_0^*	I_2	I_2	I_2
	—	I_2^*	I_2	I_2

Arr. No. 13: $xyzt(x+y)(y+z)(x-z-t)(Ax-Az+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1468, 1578, 2345, 2367, 2468, 2578, 3468, 3578, 4567, 4568, 4578, 4678, 5678

Minimilizing permutation: (1728)(364)

Symmetries: $S_3 \oplus C_2 \oplus C_2$, $\langle (13)(478)(56), (78), (15)(36), (478) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY,

Singular points:

p_4^1 : 1245, 1275, 1285, 2346, 2376, 2386, 2478

p_5^1 : 13478, 45678

p_5^2 : 12356

l_3 : 125, 236, 478

Elliptic fibrations:	∞	0	-1
	I_2^*	I_2	I_2
	—	I_0^*	I_0^*

and

∞	-1	0	1
I_4	I_2	I_4	I_2
I_0^*	—	I_0^*	—

Arr. No. 16: $xyzt(x+y)(y+z)(Ay-Bz-At)(Bx-Ay+At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 1468, 1568, 2345, 2367, 2478, 3578, 4568

Minimilizing permutation: (1352)(46)(78)

Symmetries: $C_2 \oplus C_2$, $\langle (16)(35)(78), (15)(36) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: Arr. 1,

Singular points:

p_4^0 : 1378, 5678

p_4^1 : 1275, 2386

p_5^1 : 12458, 23467

p_5^2 : 12356

l_3 : 125, 236

Elliptic fibrations:	∞	0	1	$\frac{A+B}{A}$
	I_2^*	—	I_2	I_2
	I_2	I_0^*	I_2	I_2

Arr. No. 20: $xyzt(x+y)(y+z)(x-z+t)(Ay-Bz-At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 2345, 2367, 2468, 2578,

Minimilizing permutation: (15478632)

Special values: ∞ : Arr. 3, 0: non-CY, -1: Arr. 1, -1/2: Arr. 19,

Singular points:

p_4^0 : 1347, 3578, 4567

p_4^1 : 1245, 1275, 1285, 2376

p_5^1 : 23468

p_5^2 : 12356

l_3 : 125, 236

Elliptic fibrations:	∞	0	1	$\frac{2A+B}{A}$	$-\frac{A}{B}$
	I_2	I_2	I_4	I_2	I_2
	I_2^*	I_2	I_2	—	—

Arr. No. 21: $xyzt(x+y)(y+z)(Ax-By+(-A-B)t)(Ax+Bz-At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1456, 1457, 1458, 2345, 2367, 2468, 3478, 5678

Minimilizing permutation: (1347652)

Symmetries: C_2 , $\langle(15)(47)\rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY,

Singular points:

p_4^0 : 1348, 3578, 4678

p_4^1 : 1285, 2346, 2376, 2386

p_5^1 : 12457

p_5^2 : 12356

l_3 : 125, 236

Elliptic fibrations:	∞	0	1	$-\frac{B}{A}$	$-\frac{B}{A+B}$
	I_2	I_4	I_2	I_2	I_2
	I_2^*	I_2	—	I_2	—

and

∞	0	-1	$-\frac{A}{A+B}$
I_0^*	I_2	I_2	I_2
I_2	I_2	I_0^*	I_2

Arr. No. 33: $xyzt(x+y)(y+z)(x-z+t)(Ax-Ay-Az+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1456, 1457, 1458, 1678, 2345, 2467, 2568, 3468

Minimilizing permutation: (1462)(78)

Symmetries: C_2 , $\langle(16)(35)\rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: Arr. 3, 1/2: Arr. 32,

Singular points:

p_4^0 : 1347, 1468, 2478, 4567

p_4^1 : 1245, 1275, 1285, 2346, 2376, 2386

p_5^2 : 12356

l_3 : 125, 236

Elliptic fibrations:

∞	0	-1	1	$-\frac{A}{A-B}$
I_4	I_4	I_2	I_2	$-$
I_0^*	I_2	I_2	$-$	I_2

Arr. No. 34: $xyzt(x+y)(x+z)(x+y+z+t)(Ay-Az+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1456, 1457, 1458, 2345, 2467, 2568, 3468, 3567

Minimilizing permutation: (2546)(78)

Symmetries: D_4 , $\langle(25)(78), (26)(35)\rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: Arr. 19, -1: Arr. 19,

Singular points:

p_4^0 : 2348, 2467, 3457, 4568

p_4^1 : 1245, 1275, 1285, 1346, 1376, 1386

p_5^2 : 12356

l_3 : 125, 136

Arr. No. 35: $xyzt(Ax+By)(Ax+By+At)(x+y+z+t)(By+Bz+At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1456, 1457, 1458, 2345, 2467, 2568, 3468, 3578

Minimilizing permutation: (1425)(386)

Symmetries: $C_2 \oplus C_2$, $\langle(14)(26), (12)(38)(46)\rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: Arr. 1, -1: Arr. 32

Singular points:

p_4^0 : 1368, 1478, 2348, 2367

p_4^1 : 1235, 1275, 1285, 3456, 4576, 4586

p_5^2 : 12456

l_3 : 125, 456

Arr. No. 36: $xyzt(x+y)(y-z+t)(Ax-By+Bz+At)(Ax+Ay+Bz+At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1456, 1457, 1458, 1678, 2345, 2467, 2568, 3478

Minimilizing permutation: (12)(375)(48)

Symmetries: C_2 , $\langle(18)(36)(57)\rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -1/2: Arr. 32,

Singular points:

p_4^0 : 1368, 1467, 2346, 3458
 p_4^1 : 1235, 1245, 1265, 2378, 2478, 2678
 p_5^2 : 12758
 l_3 : 125, 278

Elliptic fibrations:

∞	0	-1	$\frac{B}{A}$	$\frac{A+B}{A}$
I_4	I_2	I_2	I_2	I_2
I_0^*	I_2	I_2	I_2	—

Arr. No. 53: $xyzt(x+y)(z+t)(Ax-By-Az-At)(Bx+By-Bz+At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1468, 1478, 1678, 2345, 2467, 2568, 3467, 3578, 4567, 4678

Minimilizing permutation: (1785426)

Symmetries: $C_2 \oplus C_2$, $\langle (13)(24)(56)(78), (12)(34) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY,

Singular points:

p_4^0 : 1378, 2478
 p_4^1 : 1235, 1245, 1285, 1346, 2346, 3476
 p_5^1 : 12657, 34568
 l_3 : 125, 346

Elliptic fibrations:

∞	0	1	$-\frac{B}{A}$
I_2^*	—	I_2	I_2
—	I_2^*	I_2	I_2

and

∞	0	1	$-\frac{B}{A}$
I_0^*	I_2	I_2	I_2
I_2	I_0^*	I_2	I_2

Arr. No. 70: $xyzt(x-y+z)(y-z-t)(x-y-t)(Ax+By)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1468, 1578, 2345, 2367, 2478, 4567

Minimilizing permutation: (12)(368)(57)

Symmetries: C_2 , $\langle (37)(45) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: Arr. 3, -1/2: Arr. 69,

Singular points:

p_4^0 : 1456, 2346, 2567, 3457
 p_4^1 : 1268
 p_5^1 : 12358, 12478
 l_3 : 128

Elliptic fibrations:

∞	0	1	-1	$\frac{A}{A+B}$
I_0^*	I_2	I_2	—	I_2
I_4	I_4	I_2	I_2	—

Arr. No. 71: $xyzt(x+y)(x+y+z+t)(Ax-By+Az)(By-Az-At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1468, 1578, 2345, 2367, 2478, 3568

Minimilizing permutation: (12)(365)(48)

Symmetries: $C_2 \oplus C_2$, $\langle (15)(38)(67), (36)(78) \rangle$

Special values: ∞ : Arr. 1, 0: non-CY, -1: Arr. 1, -2: Arr. 69,

Singular points:

$$p_4^0 : 1478, 2348, 2467, 3456$$

$$p_4^1 : 1245$$

$$p_5^1 : 12357, 12658$$

$$l_3 : 125$$

Arr. No. 72: $xyzt(x+y+z)(y+z+t)(x-y-t)(Ay+Bz+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1468, 1578, 2345, 2367, 2478, 2568

Minimilizing permutation: (1837462)

Symmetries: D_4 , $\langle (26)(3745), (26)(34) \rangle$

Special values: ∞ : non-CY, 0: Arr. 19, 2: Arr. 19, 1: non-CY,

Singular points:

$$p_4^0 : 1235, 1247, 1367, 1456$$

$$p_4^1 : 1268$$

$$p_5^1 : 23468, 25678$$

$$l_3 : 268$$

Arr. No. 73: $xyzt(x+y-z-t)(y-z-t) \times$
 $\times (Ax+Ay+Bz+Bt)(Ax-By+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1267, 1345, 1367, 1468, 2345, 2367, 2478, 3568, 4567

Minimilizing permutation: (1348526)

Symmetries: C_2 , $\langle (15)(23)(78) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -2: Arr. 69,

Singular points:

$$p_4^0 : 1248, 2346, 2378, 3457$$

$$p_4^1 : 1456$$

$$p_5^1 : 12567, 13568$$

$$l_3 : 156$$

Elliptic fibrations:

∞	0	-1	$\frac{A}{B}$	$-\frac{B}{A+2B}$
I_2	I_2	I_4	I_2	I_2
<hr/>				
I_2	I_2	I_0^*	I_2	-

Arr. No. 94: $xyzt(x+y)(x+y+z-t)(Ax-By+Az)(By-Az-Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2578, 3478

Minimilizing permutation: (348675)

Special values: ∞ : non-CY, 0: non-CY, -1: Arr. 1, $\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. A, $-\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. A,

Singular points:

p_4^0 : 1368, 1478, 2348, 2467, 3456 p_4^1 : 1245, 1265, 1285 p_5^1 : 12357 l_3 : 125

Arr. No. 95: $xyzt(x+y)(x+y-z+t)(Ax-By+Bz)(Ax-By-Az-Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 3678, 4578

Minimilizing permutation: (234875)

Special values: ∞ : non-CY, 0: Arr. 3, -1: non-CY, -2: Arr. 93,

Singular points:

p_4^0 : 1368, 1467, 2468, 3456, 3478

p_4^1 : 1245, 1265, 1285

p_5^1 : 12357

l_3 : 125

Elliptic fibrations:

∞	0	-1	$\frac{B}{A}$	$-\frac{A+2B}{B}$
I_2	I_2	I_4	I_2	I_2
I_0^*	I_2	I_2	I_2	—

Arr. No. 96: $xyzt(x+y)(x+y-z+t)(Ax-By+Bz+At)(Ay+Bz+At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2578, 3678

Minimilizing permutation: (12)(368475)

Symmetries: C_2 , $\langle(12)(36)(78)\rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -1/2: Arr. 32,

Singular points:

p_4^0 : 1368, 1467, 2348, 2367, 3456

p_4^1 : 1235, 1245, 1265

p_5^1 : 12758

l_3 : 125

Elliptic fibrations:

∞	0	-1	$\frac{B}{A}$	$-\frac{A}{2A+B}$
I_2	I_2	I_4	I_2	I_2
I_0^*	I_2	I_2	I_2	—

Arr. No. 97: $xyzt(x+y)(x+y+z+t)(y-z-t)(Ax-Bz+At)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2567, 3678

Minimilizing permutation: (1365)(487)

Symmetries: $C_2 \oplus C_2$, $\langle (25)(67), (25)(48), (25)(67) \rangle$

Special values: ∞ : Arr. 19, 0: non-CY, -1: non-CY, -1/2: Arr. 93,

Singular points:

p_4^0 : 1348, 2347, 2368, 3456, 3578

p_4^1 : 1235, 1245, 1285

p_5^1 : 12657

l_3 : 125

Elliptic fibrations:

∞	0	$-\frac{1}{2}$	-1	$\frac{A}{B}$
I_0^*	I_2	—	I_2	I_2
I_2	I_4	I_2	I_4	—

Arr. No. 98: $xyzt(x+y+z)(y+z+t)(x+z-t)(Ay+Bz+Bt)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2567, 4578

Minimilizing permutation: (183562)

Symmetries: $C_2 \oplus C_2$, $\langle (17)(26), (26)(34) \rangle$

Special values: ∞ : non-CY, 0: Arr. 19, 2: Arr. 93, 1: non-CY,

Singular points:

p_4^0 : 1235, 1347, 1456, 2457, 3567

p_4^1 : 1268, 2568, 2678

p_5^1 : 23468

l_3 : 268

Elliptic fibrations:

∞	0	1	-1	$\frac{A-B}{B}$
I_4	I_4	I_2	I_2	—
I_2	I_2	—	I_0^*	I_2

Arr. No. 99: $xyzt(x+y+z)(x+z-t) \times$

$\times (Ax + (A+B)y - Bz + Bt)(Ax - By - Bz)$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 2567, 3478

Minimilizing permutation: (1247835)

Symmetries: C_2 , $\langle (15)(47) \rangle$

Special values: ∞ : non-CY, 0: Arr. 19, -1: non-CY, -2: Arr. 19,

Singular points:

p_4^0 : 1267, 1346, 2456, 2478, 3567

p_4^1 : 1458, 1568, 1578

p_5^1 : 12358

l_3 : 158

Arr. No. 100: $xyzt(x+y-z+t)(Ax+Ay+Bz) \times$

$$\times (Ay+Bz+At)(By-Bz-At)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1245, 1345, 1467, 1568, 2345, 2468, 3567, 4578

Minimilizing permutation: (167)(2358)

Symmetries: C_2 , $\langle (28)(34)(56) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -2: Arr. 69,

Singular points:

$$p_4^0 : 1236, 1357, 1458, 1467, 3456$$

$$p_4^1 : 1278, 2578, 2678$$

$$p_5^1 : 23478$$

$$l_3 : 278$$

Elliptic fibrations:

∞	0	-1	$\frac{B}{A}$	$-\frac{A+2B}{B}$
I_2	I_2	I_4	I_2	I_2
I_2	I_0^*	I_2	I_2	—

Arr. No. 144: $xyzt(x-y+z+t)(Ax+By+Az) \times$

$$\times (By+Az+At)(Bx-By-Az+Bt)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1456, 1457, 1467, 1567, 2458, 2678, 3468, 3578, 4567

Minimilizing permutation: (175328)

Symmetries: D_4 , $\langle (35)(67), (17)(46) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -2: Arr. 19,

Singular points:

$$p_4^0 : 1236, 1257, 2347, 2456$$

$$p_4^1 : 1358, 2358, 3458, 3568, 3578$$

$$p_5^0 : 14678$$

$$l_3 : 358$$

Elliptic fibrations:

∞	0	-1	-2	$\frac{A}{B}$
I_4	I_2	I_4	I_2	—
I_2	—	I_2	—	I_2^*

Arr. No. 152: $xyzt(x+y+z+t)(y+t) \times$

$$\times (x-y-z+t)(Ax-Ay+Bz-Bt)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1456, 1478, 2457, 2468, 3458, 3567

Minimilizing permutation: (1742)(36)

Symmetries: C_2 , $\langle (13)(24) \rangle$

Special values: ∞ : Arr. 32, 0: Arr. 32, 1: Arr. 19, -1: non-CY,

Singular points:

$$p_4^0 : 1278, 1356, 1457, 2357, 3478, 5678$$

$$p_4^1 : 1246, 2346, 2456, 2476, 2486$$

$$l_3 : 246$$

Arr. No. 153: $xyzt(x+y+z)(y+z+t) \times$

$$\times (Ax - By + At)(Ax - By + Az + At)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1456, 1478, 2457, 2468, 3458, 5678

Minimilizing permutation: (1824573)

Symmetries: $C_2 \oplus C_2$, $\langle (14)(56), (14)(38) \rangle$

Special values: ∞ : Arr. 19, 0: non-CY, -1: Arr. 3, -1/2: Arr. 93,

Singular points:

$$p_4^0 : 1235, 1247, 1268, 1456, 2346, 2458$$

$$p_4^1 : 1378, 2378, 3478, 3578, 3678$$

$$l_3 : 378$$

Elliptic fibrations:

∞	0	-1	-2	$\frac{B}{A}$
I_4	I_2	I_4	I_2	—
I_2	I_2	I_2	—	I_0^*

Arr. No. 154: $xyzt(x+y+z)(x+y+z-t) \times$

$$\times (Ax + (A+B)y - Bz + Bt)(Ax - Bz - At)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1456, 1478, 2457, 2568, 3458, 3678

Minimilizing permutation: (17634285)

Symmetries: C_2 , $\langle (17)(38)(45) \rangle$

Special values: ∞ : Arr. 1, 0: non-CY, -1: non-CY, -2: Arr. 32,

Singular points:

$$p_4^0 : 1235, 1267, 1348, 2368, 2478, 3578$$

$$p_4^1 : 1456, 2456, 3456, 4576, 4586$$

$$l_3 : 456$$

Arr. No. 155: $xyzt(Ax + By + Az)(Ax + (A+B)y - Bz + At) \times$

$$\times (Ax - Bz - Bt)(Ax + By + Az + At)$$

Minimal incidences: 1234, 1235, 1236, 1237, 1238, 1456, 1478, 2457, 2568, 3468, 3578

Minimilizing permutation: (174)(2538)

Symmetries: S_3 , $\langle (16)(37)(45), (237)(485) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY,

$$\frac{\sqrt{-3}}{2} - \frac{1}{2}: \text{Arr. A}, \quad -\frac{\sqrt{-3}}{2} - \frac{1}{2}: \text{Arr. A},$$

Singular points:

$$p_4^0 : 1235, 1278, 1347, 2368, 2467, 3567$$

$$p_4^1 : 1458, 2458, 3458, 4568, 4578$$

$$l_3 : 458$$

Arr. No. 197: $xyzt(x-y-z+t)(Ax + By + Bz) \times$

$$\times (By + Bz + At)(Ax + Bz + At)$$

Minimal incidences: 1234, 1235, 1245, 1267, 1345, 1368, 1478, 2345, 2378, 2468, 5678

Minimilizing permutation: (1287364)

Symmetries: $C_2 \oplus C_2$, $\langle (38)(67), (14)(67), (38)(67) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: Arr. 3, -2: Arr. 93,

Singular points:

p_4^0 : 1236, 1278, 1348, 2347, 2358, 2468

p_5^0 : 14567

Elliptic fibrations:	∞	0	1	$-\frac{B}{A}$	$-\frac{2B}{A}$
	I_4	I_2	—	I_4	I_2
	I_2	I_2	I_0^*	I_2	—

Arr. No. 198: $xyzt(x+y+z)(y+z+t)(x-y-t)(Ax-Ay-Az+Bt)$

Minimal incidences: 1234, 1235, 1245, 1267, 1345, 1368, 1478, 2345, 2378, 2468, 3567

Minimilizing permutation: (1285436)

Symmetries: C_2 , $\langle(16)(37)\rangle$

Special values: ∞ : Arr. 69, 0: non-CY, -1: Arr. 19, -1/2: Arr. 69,

Singular points:

p_4^0 : 1235, 1247, 1367, 2346, 2567, 3478

p_5^0 : 14568

Arr. No. 199: $xyzt(x+y+z)(y+z+t) \times$

$\times (Ax+By+(A-B)z)(Ax+By+Az+Bt)$

Minimal incidences: 1234, 1235, 1245, 1267, 1345, 1368, 1478, 2345, 2378, 2568, 4567

Minimilizing permutation: (154732)

Symmetries: C_2 , $\langle(15)(23)(46)\rangle$

Special values: ∞ : non-CY, 0: non-CY, 2: Arr. 69, 1: Arr. 1,

Singular points:

p_4^0 : 1368, 1456, 2346, 2458, 2678, 3478

p_5^0 : 12357

Arr. No. 200: $xyzt(x+y+z+t)(Ax+Ay-Bz-Bt) \times$

$\times (Ay-Bz+At)(Ax-By-Bt)$

Minimal incidences: 1234, 1235, 1245, 1267, 1345, 1368, 1478, 2345, 2568, 3578, 4567

Minimilizing permutation: (268)(35)(47)

Symmetries: S_3 , $\langle(13)(26)(58), (13)(46)(57)\rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, $\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. A,

$\frac{-\sqrt{-3}}{2} - \frac{1}{2}$: Arr. A,

Singular points:

p_4^0 : 1248, 1256, 1467, 2347, 2368, 3456

p_5^0 : 13578

Arr. No. 242: $xyzt(x+y+z)(x+z-t) \times$

$\times (Ax+(A+B)y-Bz+Bt)((A+B)x+(A+B)y+Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2457, 2468, 3456, 3478, 5678

Minimilizing permutation: (147685)

Symmetries: $G_{64,138}$, $\langle(1658)(2437), (1253)(4876), (1736)(2854)\rangle$

Special values: ∞ : non-CY, 0: Arr. 238, -1: non-CY, -2: Arr. 238,

Singular points:

p_4^0 : 1235, 1248, 1267, 1346, 1378, 2456, 2578, 3458, 3567, 4678

Elliptic fibrations:

∞	0	1	-1	$\frac{A+B}{B}$	$-\frac{A+B}{B}$
I_4	I_4	I_2	I_2	—	—
I_4	I_4	—	—	I_2	I_2

and

∞	0	-1	$-\frac{A+B}{B}$	$-\frac{A+2B}{B}$
I_4	I_2	I_2	I_2	I_2
I_4	I_2	I_2	I_2	I_2

Arr. No. 243: $xyzt(x+y+z)(y+z+t)(x+y+t)(Ax+By+Az+At)$ **Minimal incidences:** 1234, 1256, 1278, 1357, 1368, 1458, 2367, 2457, 3456**Minimilizing permutation:** (157642)**Symmetries:** S_3 , $\langle(34)(57), (13)(67)\rangle$ **Special values:** ∞ : Arr. 239, 0: non-CY, 1: Arr. 3, $1/2$: Arr. 238,
 $2/3$: Arr. 240,**Singular points:** p_4^0 : 1235, 1247, 1268, 1367, 1456, 2346, 2378, 2458, 3457**Elliptic fibrations:**

∞	0	-1	-2	$-\frac{A}{2A-B}$	$-\frac{B}{A}$
I_2	I_2	I_4	—	I_2	I_2
I_4	I_2	I_4	I_2	—	—

Arr. No. 244: $xyzt(x+y+z+t)(Ax+Ay+Bz+Bt) \times$
 $\times (Ay+Bz+At)(Ax+Bz+At)$ **Minimal incidences:** 1234, 1256, 1278, 1357, 1368, 2457, 2468, 3458, 5678**Minimilizing permutation:** (154)(2863)**Symmetries:** $C_2 \oplus C_2$, $\langle(12)(34)(56)(78), (34)(56)\rangle$ **Special values:** ∞ : non-CY, 0: non-CY, 2: Arr. 240, 1: non-CY,
 $1/2$: Arr. 240,**Singular points:** p_4^0 : 1256, 1278, 1348, 1357, 1467, 2347, 2358, 2468, 3456**Elliptic fibrations:**

∞	0	-1	$\frac{B^2}{A(A-2B)}$	$-\frac{B(2A-B)}{A^2}$	$-\frac{B}{A}$
I_2	I_2	I_2	I_2	—	I_4
I_2	I_2	I_2	—	I_2	I_4

and

∞	0	$-\frac{1}{2}$	-1	$-\frac{A}{B}$	$-\frac{B}{A}$
I_2	I_4	I_2	I_4	—	—
I_2	I_2	—	I_4	I_2	I_2

Arr. No. 246: $xyzt(x+y+z)(Ax+(A+B)y-Bz+Bt) \times$
 $\times (Ax-Bz-At)(Ax+(A+B)y+Az-At)$
Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 2467, 3678, 4578
Minimilizing permutation: (16327)(45)
Symmetries: $D_4 \oplus C_2$, $\langle (23)(46)(78), (14)(37)(56), (14)(28)(56) \rangle$
Special values: ∞ : Arr. 1, 0: non-CY, -1: non-CY, -2: Arr. 241,
Singular points:
 p_4^0 : 1235, 1268, 1347, 1578, 2378, 2458, 2467, 3468, 3567

Arr. No. 247: $xyzt(x+y+z)(y+z+t)(x-y-t)(Ax-Bz+Bt)$
Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2367, 5678
Minimilizing permutation: (267584)
Symmetries: D_4 , $\langle (26)(34), (37)(45) \rangle$
Special values: ∞ : non-CY, 0: Arr. 93, -1: Arr. 238, -2: Arr. 93,
Singular points:

p_4^0 : 1235, 1247, 1348, 1367, 1456, 1578, 2346, 2567
Elliptic fibrations:

∞	0	1	-1	$-\frac{B}{A+B}$	$-\frac{A+B}{B}$
I_4	I_4	I_2	I_2	—	—
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 248: $xyzt(x+y+z)(y-z-t)(x+z+t) \times$
 $\times (Ax+(A+B)y-Bz+At)$
Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2358, 2367
Minimilizing permutation: (17)(23)(48)(56)
Symmetries: $C_2 \oplus C_2$, $\langle (15)(23), (15)(46) \rangle$
Special values: ∞ : Arr. 239, 0: Arr. 19, -1: non-CY, -2: Arr. 245,
-1/2: Arr. 239, -2/3: Arr. 245,
Singular points:

p_4^0 : 1235, 1267, 1347, 2346, 2378, 2457, 3567, 4678
Elliptic fibrations:

∞	0	1	-1	$\frac{A+B}{B}$	$\frac{2A+B}{A+B}$
I_4	I_4	I_2	I_2	—	—
I_2	I_2	I_4	—	I_2	I_2

Arr. No. 249: $xyzt(x+y+z)(x+z+t) \times$
 $\times (Ax+(A+B)y-Bz+At)(By-Bz+At)$
Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 2467, 3456
Minimilizing permutation: (183)(57)
Symmetries: $C_2 \oplus C_2 \oplus C_2$, $\langle (15)(47)(68), (15)(23), (23)(48)(67) \rangle$
Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -2: Arr. 241,
Singular points:
 p_4^0 : 1235, 1278, 1346, 2348, 2367, 2456, 3578, 4678

Elliptic fibrations:

∞	0	1	-1	$\frac{B}{A+B}$	$\frac{A+B}{B}$
I_4	I_4	I_2	I_2	—	—
I_2	I_2	I_4	—	I_2	I_2

Arr. No. 250: $xyzt(x+y+z)(y+z-t)(x+z+t)(Ax+By-Az+At)$ **Minimal incidences:** 1234, 1256, 1278, 1357, 1368, 2358, 2467, 5678**Minimilizing permutation:** (173)(458)**Symmetries:** C_2 , $\langle(16)(45)\rangle$ **Special values:** ∞ : Arr. 69, 0: non-CY, 1: Arr. 245, -1: Arr. 93,
-1/2: Arr. 240,**Singular points:** p_4^0 : 1235, 1268, 1347, 1456, 2346, 2378, 2457, 3567**Elliptic fibrations:**

∞	0	1	-1	$-\frac{B}{A}$	$-\frac{B}{2A+B}$
I_4	I_4	I_2	I_2	—	—
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 251: $xyzt(x+y+z)(x+z-t) \times$ $\times (Ax + (A+B)y - Bz + Bt)(Ax - By - Bz - At)$ **Minimal incidences:** 1234, 1256, 1278, 1357, 1368, 2358, 2467, 4568**Minimilizing permutation:** (1846)**Special values:** ∞ : Arr. 1, 0: Arr. 19, -1: non-CY, -2: Arr. 93,
 $-\frac{\sqrt{5}}{2} - \frac{1}{2}$: Arr. C, $\frac{\sqrt{5}}{2} - \frac{1}{2}$: Arr. C,**Singular points:** p_4^0 : 1235, 1267, 1346, 1458, 2368, 2456, 2478, 3567**Arr. No. 252:** $xyzt(x+y+z)(x+y+t) \times$ $\times (Ax + 2Ay - Bz + At)(Ax - Bz - At)$ **Minimal incidences:** 1234, 1256, 1278, 1357, 1468, 3456, 3678, 4578**Minimilizing permutation:** (152643)(78)**Symmetries:** $C_2 \oplus C_2$, $\langle(14)(26), (14)(26), (16)(24)(78)\rangle$ **Special values:** ∞ : non-CY, 0: non-CY, -1: Arr. 69, -1/2: Arr. 241,**Singular points:** p_4^0 : 1235, 1246, 1348, 1678, 2367, 2478, 3456, 3578**Elliptic fibrations:**

∞	0	-1	-2	$\frac{B}{A}$	$-\frac{2A+B}{A}$
I_4	I_2	I_4	I_2	—	—
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 253: $xyzt(x+y+z)(x+z-t) \times$
 $\times (Ax + (A+B)y - Bz + Bt)(Ax + Ay - Bz - At)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2358, 2457, 3467

Minimilizing permutation: (18476)(23)

Special values: ∞ : Arr. 3, 0: non-CY, -1: non-CY, -2: Arr. 240,
 $-1/2$: Arr. 245, $\frac{\sqrt{5}}{2} - \frac{3}{2}$: Arr. C, $-\frac{\sqrt{5}}{2} - \frac{3}{2}$: Arr. C,

Singular points:

p_4^0 : 1235, 1267, 1346, 2368, 2456, 2478, 3458, 3567

Elliptic fibrations:

∞	0	1	$-\frac{A}{B}$	$-\frac{A+B}{B}$	$\frac{2A+B}{A}$
I_4	I_2	I_2	I_2	I_2	—
I_2	I_2	I_4	I_2	—	I_2

Arr. No. 254: $xyzt(x+y+z+t)(Ax + Ay - Bz - Bt) \times$
 $\times (Ay - Bz + At)(Ax - By - Bz)$

Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 3456, 3678

Minimilizing permutation: (265)(384)

Symmetries: C_2 , $\langle (14)(23)(56)(78) \rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY, -2: Arr. 241,
 $\frac{\sqrt{5}}{2} - \frac{3}{2}$: Arr. C, $-\frac{\sqrt{5}}{2} - \frac{3}{2}$: Arr. C,

Singular points:

p_4^0 : 1238, 1256, 1357, 1458, 1467, 2347, 2468, 3456

Elliptic fibrations:

∞	0	-1	$\frac{B}{A}$	$\frac{B^2}{A(A+2B)}$	$-\frac{A+2B}{B}$
I_2	I_2	I_2	I_4	I_2	—
I_2	I_2	I_4	I_2	—	I_2

Arr. No. 255: $xyzt(Ax + Ay + Bz + Bt)(x + y - 2z - 2t) \times$
 $\times (Ay + (-2A + B)z + Bt)(Bx + (-2A + B)y + (4A - 2B)z - 2Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2457, 3458, 4678

Minimilizing permutation: (15382746)

Special values: ∞ : non-CY, 0: non-CY, $1/2$: Arr. 32,
 $-1/2$: non-CY, $\frac{\sqrt{5}}{4} + \frac{1}{4}$: Arr. C, $-\frac{\sqrt{5}}{4} + \frac{1}{4}$: Arr. C,

Singular points:

p_4^0 : 1256, 1278, 1357, 1468, 2347, 2368, 3456, 5678

Elliptic fibrations:

∞	0	$\frac{1}{2}$	$-\frac{A}{B}$	$-\frac{2A-B}{2B}$	$\frac{A(2A-B)}{B^2}$
I_4	I_2	I_2	I_2	I_2	—
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 256: $xyzt(x+y+2z)(Ay-Bz+Bt) \times$
 $\times (Ax+Ay+(2A-B)z+Bt)(Bx+(-2A+B)y+2Bz-2Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2358, 2457, 3456

Minimilizing permutation: (17462)(35)

Symmetries: D_4 , $\langle (17)(25)(36)(48), (1847)(2536), (1847)(2536) \rangle$

Special values: ∞ : non-CY, 0: non-CY, $1/2$: Arr. 238, $-1/2$: Arr. 239,
 $\frac{\sqrt{-3}}{4} + \frac{1}{4}$: Arr. B, $-\frac{\sqrt{-3}}{4} + \frac{1}{4}$: Arr. B,

Singular points:

p_4^0 : 1235, 1268, 1367, 2346, 2458, 2567, 3457, 3568

Elliptic fibrations:

∞	0	1	$-\frac{2A}{B}$	$-\frac{2A-B}{B}$	$\frac{2A}{2A-B}$
I_4	I_2	I_2	I_2	I_2	—
	└──────────┘				
I_2	I_4	I_2	I_2	—	I_2
└────────┘		└──────────┘			

Arr. No. 257: $xyzt(x+y+2z+2t)(Ax+Ay+Bz+Bt) \times$
 $\times (Ay+(-2A+B)z+Bt)((2A-B)x-By+(4A-2B)z-2Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2457, 3456, 4678

Minimilizing permutation: (15)(2748)(36)

Symmetries: C_2 , $\langle (23)(68) \rangle$

Special values: ∞ : non-CY, 0: non-CY, $1/2$: non-CY, $1/4$: Arr. 240,
 $\frac{\sqrt{-3}}{4} + \frac{1}{4}$: Arr. B, $-\frac{\sqrt{-3}}{4} + \frac{1}{4}$: Arr. B,

Singular points:

p_4^0 : 1256, 1278, 1358, 1367, 2347, 2458, 3456, 5678

Elliptic fibrations:

∞	0	-1	$\frac{2A-B}{B}$	$-\frac{2A}{B}$	$\frac{2A(2A-B)}{B^2}$
I_4	I_2	I_2	I_2	I_2	—
	└────────┘		└────────┘		
I_2	I_2	I_2	I_2	I_2	I_2
└────────┘		└──────────┘			

Arr. No. 258: $xyzt(x-y+2z-2t)(y-z+2t) \times$
 $\times (x-y+z-t)(Ax+By+Az+Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2367, 4567

Minimilizing permutation: (1367)(245)

Symmetries: C_2 , $\langle (15)(36) \rangle$

Special values: ∞ : Arr. 32, 0: Arr. 93,
-1: non-CY, -2: Arr. 245, $-1/2$: Arr. 240,

Singular points:

p_4^0 : 1257, 1356, 1378, 1467, 2346, 2478, 3457, 5678

Elliptic fibrations:

∞	0	1	$\frac{1}{2}$	$-\frac{A}{B}$	$-\frac{A}{2B}$
I_4	I_2	I_2	I_4	—	—
	└────────┘		└────────┘		
I_2	I_2	I_2	I_2	I_2	I_2
└────────┘		└──────────┘			

Arr. No. 259: $xyzt(x+y+z+t)(x-y-z+t) \times$
 $\times (Ax - Ay + Bz - Bt)(Ax - By + Az - Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2457, 3458, 3467

Minimilizing permutation: (1825)(3746)

Symmetries: $C_2 \oplus C_2$, $\langle (14)(56)(78), (14)(23) \rangle$

Special values: ∞ : Arr. 32, 0: Arr. 32, 1: non-CY, -1: non-CY,

Singular points:

p_4^0 : 1267, 1358, 1456, 2356, 2378, 2458, 3467, 5678

Elliptic fibrations:

∞	0	1	-1	$\frac{A}{B}$	$-\frac{A}{B}$
I_4	-	I_2	I_2	I_2	I_2
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 261: $xyzt(x+y+z+t)(x-y-z+t) \times$
 $\times (Ax - Ay + Bz - Bt)(Ax + Ay + Bz + Bt)$

Minimal incidences: 1234, 1256, 1357, 1467, 2358, 2468, 3678, 4578

Minimilizing permutation: (16245)(78)

Symmetries: $D_4 \oplus C_2$, $\langle (14)(23), (56)(78), (1324)(58)(67) \rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: non-CY, -1: non-CY,

Singular points:

p_4^0 : 1258, 1267, 1378, 1456, 2356, 2478, 3458, 3467

Elliptic fibrations:

∞	0	1	-1	$\frac{A}{B}$	$-\frac{A}{B}$
I_2	I_2	I_2	I_2	I_2	I_2
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 262: $xyzt(x-z-t)(Ax + Ay + Bz) \times$
 $\times (Ax + (A+B)y - Az + Bt)(By + (-A-B)z - At)$

Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 3467, 4578

Minimilizing permutation: (163485)(27)

Special values: ∞ : Arr. 1, 0: non-CY, -1: non-CY, $\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. A,
 $-\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. A,

Singular points:

p_4^0 : 1236, 1345, 1578, 2348, 2457, 2568, 3567, 4678

Arr. No. 264: $xyzt(y-2z+2t)(Ax + Ay + Bz) \times$
 $\times (Ax + 2Ay + (-2A+B)z + (2A-B)t)(Ax + Ay - 2Az + (2A-B)t)$

Minimal incidences: 1234, 1256, 1278, 1357, 2468, 3456, 3678, 4578

Minimilizing permutation: (1384275)

Symmetries: D_4 , $\langle (1678)(24)(35), (17)(25)(34), (25)(34)(68) \rangle$

Special values: ∞ : non-CY, 0: non-CY, 1/2: non-CY, -1/2: non-CY,

Singular points:

p_4^0 : 1236, 1257, 1458, 1678, 2345, 2378, 3468, 4567

Elliptic fibrations:

∞	0	$-\frac{1}{2}$	$\frac{A}{B}$	$-\frac{2A}{2A-B}$	$-\frac{A}{2A-B}$
I_2	I_2	I_2	I_2	I_2	I_2
I_2	I_2	I_2	I_2	I_2	I_2

and

∞	0	1	$\frac{2A+B}{2A-B}$	$\frac{2A}{2A-B}$	$\frac{B^2+4A^2}{2A(2A-B)}$
I_4	—	I_2	I_2	I_4	—
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 265: $xyzt(x+y-z+2t)(Ax+2Ay-Az+Bt) \times$
 $\times (By-2Az+2Bt)(Bx+By+(2A-B)z)$

Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 2467, 3478

Minimilizing permutation: (16587)(34)

Symmetries: $C_2 \oplus C_2$, $\langle (27)(58), (16)(27)(34) \rangle$

Special values: ∞ : non-CY, 0: non-CY, $1/2$: non-CY, $1/4$: Arr. 69,

Singular points:

p_4^0 : 1238, 1267, 1357, 2347, 2456, 2578, 3458, 4678

Elliptic fibrations:

∞	0	$-\frac{1}{2}$	$\frac{A}{2A-B}$	$-\frac{2A}{B}$	$-\frac{A}{B}$
I_2	I_2	I_2	I_2	I_2	I_2
I_2	I_4	I_2	I_2	—	I_2

Arr. No. 266: $xyzt(y-2z+2t)(2x+y+2t) \times$
 $\times (Ax+By+Az)(Ax+(A+B)y-Az+At)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 1458, 2358, 2467

Minimilizing permutation: (152)(68)

Symmetries: S_3 , $\langle (163)(487), (36)(47) \rangle$

Special values: ∞ : Arr. 19, 0: non-CY, 2: Arr. 245, -1: Arr. 245,
-2: Arr. 19, -4: Arr. 245,

Singular points:

p_4^0 : 1237, 1246, 1258, 1356, 2345, 2368, 2567, 4578

Arr. No. 267: $xyzt(Ax+Ay+(-A+B)z)(Ax+By-Az+At) \times$
 $\times ((-A+B)y-Bz+Bt)(Bx+By-Az+Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 2468, 3458, 3678, 4567

Minimilizing permutation: (147853)

Symmetries: S_3 , $\langle (176)(485), (15)(23)(46)(78) \rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: non-CY,
 $\frac{\sqrt{-3}}{2} + \frac{1}{2}$: non-CY, $-\frac{\sqrt{-3}}{2} + \frac{1}{2}$: non-CY,

Singular points:

p_4^0 : 1235, 1267, 1468, 1578, 2347, 2368, 3458, 4567

Elliptic fibrations:

∞	0	-1	$\frac{A-B}{B}$	$-\frac{A}{A-B}$	$-\frac{B}{A}$
I_2	I_2	I_2	I_2	I_2	I_2
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 268: $xyzt(x+y+z)(Ay-2Bz+2Bt) \times$
 $\times (2Bx+2By+At)((-A+2B)x+2By-Az+At)$

Minimal incidences: 1234, 1256, 1278, 1357, 1468, 2358, 3467, 5678

Minimilizing permutation: (12)(38)(46)(57)

Special values: p_4^0 : 1235, 1247, 1268, 1378, 2346, 2578, 3457, 4568

Singular points:

Elliptic fibrations: ∞ : non-CY, 0: non-CY, 2: non-CY, -2: Arr. 69,
 $\sqrt{5}-1$: Arr. C, $-\sqrt{5}-1$: Arr. C,

∞	0	1	$\frac{A-2B}{A}$	$-\frac{A}{2B}$	$-\frac{2B}{A}$
I_2	I_2	I_2	I_2	I_2	I_2
I_4	I_2	I_2	I_2	-	I_2

Arr. No. 270: $xyzt(x+y+z)(y+z+t) \times$
 $\times (Ax+2Ay-Bz+At)(Bx-2Ay+Bz+Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2458, 3467, 5678

Minimilizing permutation: (182)(4567)

Symmetries: C_4 , $\langle(1645)(38)\rangle$

Special values: ∞ : non-CY, 0: non-CY, -1: non-CY,
 -1/2: non-CY,

Singular points:

p_4^0 : 1235, 1268, 1456, 1478, 2346, 2378, 2458, 3567

Elliptic fibrations:

∞	0	-1	-2	$\frac{B}{A}$	$\frac{2A}{B}$
I_4	I_2	I_4	I_2	-	-
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 273: $xyzt(x+y+z)(2y+2z+t) \times$
 $\times (2x-2z-t)(Ax+2By-Az+Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2358, 2367, 4567

Minimilizing permutation: (1258476)

Symmetries: S_3 , $\langle(16)(27), (163)(247)\rangle$

Special values: ∞ : Arr. 19, 0: Arr. 19, 2: Arr. 245, -1: Arr. 245,
 -2: Arr. 19, -4: Arr. 245,

Singular points:

p_4^0 : 1235, 1267, 1347, 1368, 1456, 2346, 2478, 3567

Arr. No. 274: $xyzt(x+y+z)(x+z-t) \times$
 $\times (Ax + (A+B)y - Bz + Bt)(Ax + Ay - Bz + (A+B)t)$

Minimal incidences: 1234, 1256, 1278, 1357, 1368, 2358, 2457, 4678

Minimilizing permutation: (13528476)

Symmetries: C_2 , $\langle (13)(24)(56)(78) \rangle$

Special values: ∞ : Arr. 3, 0: Arr. 3, -1: non-CY, -2: Arr. 245,
 $-1/2$: Arr. 245, $\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. B, $-\frac{\sqrt{-3}}{2} - \frac{1}{2}$: Arr. B,

Singular points:

p_4^0 : 1235, 1267, 1346, 1568, 2456, 2478, 3458, 3567

Elliptic fibrations:

∞	0	1	$-\frac{A}{B}$	$-\frac{A+B}{B}$	$-\frac{A}{A+B}$
I_4	I_2	I_2	I_2	I_2	—
I_2	I_4	I_2	I_2	—	I_2

Arr. No. 275: $xyzt(Bx + (A+B)y + (A-B)z)(2Bx + (2A+2B)z + Bt) \times$
 $\times (8By + (4A-4B)z + (A-B)t)((2A-2B)x + (-4A-4B)y + (A+B)t)$

Minimal incidences: 1234, 1256, 1357, 1467, 2358, 2478, 3678, 4568

Minimilizing permutation: (123584)

Symmetries: C_6 , $\langle (173826)(45) \rangle$

Special values: ∞ : non-CY, 1: non-CY, -1: non-CY,
 $\sqrt{-3}$: non-CY, $-\sqrt{-3}$: non-CY,

Singular points:

p_4^0 : 1235, 1248, 1346, 1578, 2347, 2567, 3568, 4678

Elliptic fibrations:

∞	0	$-\frac{2B}{A-B}$	$-\frac{2B}{A+B}$	$\frac{B^2}{(A-B)^2}$	$-\frac{A+B}{A-B}$
I_2	I_2	I_2	I_2	I_2	I_2
I_2	I_2	I_2	I_2	I_2	I_2

Arr. No. 276: $xyzt(x+z+t)(Ax + By + Bt)((-A+B)x - By - Az)(-Ay - Az + (-A+B)t)$

Minimal incidences: 1234, 1256, 1357, 1468, 2358, 2478, 3678, 4567

Minimilizing permutation: (1738)(265)

Symmetries: C_8 , $\langle (15274638) \rangle$

Special values: ∞ : non-CY, 0: non-CY, 1: non-CY, -1: non-CY

Singular points:

p_4^0 : 1237, 1246, 1345, 1678, 2348, 2568, 3567, 4578

Arr. No. D: $xyzt(x+y+z+t)((\sqrt{-3}+1)x + (\sqrt{-3}+1)y + 2z)(Ax + (1-\sqrt{-3})By + 2Bt) \times$
 $\times (((1-\sqrt{-3})A - (1-\sqrt{-3})^2B)x + 4Bz + 4Bt)$

Minimal incidences: 1234, 1256, 1278, 1357, 1468, 3458, 3678, 4567

Minimilizing permutation: (384675)

Symmetries: S_3 , $\langle (37)(46)(58), (35)(47)(68) \rangle$

Special values: ∞ : non-CY, 0: Arr. A, 2: Arr. A, $1-\sqrt{-3}$: non-CY

Singular points:

p_4^0 : 1236, 1247, 1258, 1348, 1567, 3456, 3578, 4678

6.3. Applications. Birational transformation between two versions of the surfaces of type S_3 described in the section 4 was used in [4] to prove that the double octic Calabi–Yau threefolds no. 32 and 69 are birational. Using this birational transformation and a similar transformation for surface S_6 together with the elliptic surface fibration we found thirteen pairs of birational double octics.

Self isogeny of surface of type S_2 (swapping I_2 and I_4 fibers) and quadratic pullbacks of S_1 , S_3 and S_4 gives several examples of correspondences between double octics.

Theorem 6.2. *The following pairs of double octic Calabi–Yau threefolds are birational (32, 69), (10, 16), (21, 53), (33, 70), (36, 73), (96, 100), (97, 98), (153, 197), (250, 258), (259, 265), (255, 268), (261, 264), (267, 275).*

Theorem 6.3. *There exist correspondences between the following pairs of double octic Calabi–Yau threefolds (1, 238), (32, 93), (238, 241), (240, 245), (2, 242), (8, 249), (10, 242), (247, 252).*

From the incidence tables used in the classification algorithm we have derived the groups of permutation of planes that preserves the incidences. With simple linear algebra one can check which symmetries correspond to actual automorphisms of the Calabi–Yau threefold. For any of eleven rigid double octics defined over \mathbb{Q} the answer is always yes, as it was already verified in [15] for the arrangement no. 238 (the one with the largest symmetry group). In remaining cases the situation is more complicated, invariant permutation may correspond to an isomorphism with another member of family (in the case of a family defined over \mathbb{Q}), isomorphism with the Galois conjugate example (arrangements A, B, C) or element of the conjugate family (arrangement D). In fact all the possible phenomena occurred, since discussion of all examples is beyond the scope of current paper we shall only give some examples check if the two Galois–conjugate arrangements of types A, B, C, D are projectively equivalent and list examples of maximal automorphisms (cf. [18]).

Theorem 6.4. *The Galois–conjugate arrangements of type A, B or D are projectively equivalent, the Galois–conjugate arrangements of type C are not projectively equivalent.*

Proof. Permutation $(2, 5)(3, 7)(4, 6)$ corresponds to the following automorphism

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \begin{pmatrix} (\sqrt{-3} - 1)x \\ (-\sqrt{-3} + 1)(x + y) \\ (-\sqrt{-3} - 1)x - 2y + (-\sqrt{-3} - 1)z \\ (-\sqrt{-3} - 1)(x + y + z - t) \end{pmatrix}$$

of the projective space \mathbb{P}^3 which transforms the arrangement of type A to its Galois conjugate.

Permutation $(13)(24)(56)(78)$ corresponds to the following automorphism

$$(x, y, z, t) \mapsto (z, -t, x, -y)$$

of the projective space \mathbb{P}^3 which transforms the arrangement of type A to its Galois conjugate.

Permutation $(3, 5), (4, 7), (6, 8)$ corresponds to the following automorphism

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \begin{pmatrix} 4Bx \\ (-2B\sqrt{-3} + A\sqrt{-3} + 2B - A)y \\ (2A - 4B)x + (2A - 4B)y + (2A - 4B)z + (2A - 4B)t \\ -2Ax + (-A\sqrt{-3} - A + 2B + 2B\sqrt{-3})y + (-2A + 4B)t \end{pmatrix}$$

of the projective space \mathbb{P}^3 which transforms the fiber $(A : B)$ of first family to the fiber over $(2A : A - 2B)$ of the conjugate family.

The only non-trivial symmetry of arrangements of type C is $((1, 7), (2, 5), (3, 8), (4, 6))$ and it corresponds to the projective transformation

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \begin{pmatrix} 2x + 2y + (\sqrt{5} - 1)t \\ (-\sqrt{5} + 1)x + (-\sqrt{5} + 1)y + (-\sqrt{5} + 1)z \\ (\sqrt{5} - 3)x - 2y + (\sqrt{5} - 1)z + (-\sqrt{5} + 1)t \\ (-\sqrt{5} + 1)y + 2z - 2t \end{pmatrix}$$

which maps an arrangement of type C onto the same octic arrangement, \square

Rohde ([18]) used the notion of maximal automorphism of a Calabi–Yau threefolds to study density of CM-points and give examples of families of Calabi–Yau threefolds without a point of Maximal Unipotent Monodromy

Definition 6.1. A maximal automorphism of a family of Calabi–Yau threefolds is an automorphism of a smooth fiber which extends to the local universal deformation space.

Proposition 6.5. *One parameter families of double octic Calabi–Yau threefolds defined by arrangements No. No. 4, 13, 34, 72, 261 have maximal automorphisms acting on $H^{3,0}$ as multiplication by i .*

Proof. We give an example of maximal automorphism for each of the above arrangements, we also list the corresponding permutation of planes

Arr. No. 4 $((1, 5), (3, 6), (7, 8))$

$$(x, y, z, t, u) \mapsto (x + y, -y, y + z, t, iu)$$

Arr. No. 13 $((1, 3), (5, 6))$

$$(x, y, z, t, u) \mapsto (z, y, x, -t, -iu)$$

Arr. No. 34 $((2, 5), (3, 6))$

$$(x, y, z, t, u) \mapsto (x, -x - y, -z - x, -t, iu)$$

Arr. No. 72 $((3, 4), (5, 7))$

$$(x, y, z, t, u) \mapsto (x, -y, -t, -z, -iu)$$

Arr. No. 261 $((1, 3, 2, 4), (5, 8), (6, 7))$

$$(x, y, z, t, u) \mapsto (Bz, Bt, Ay, Ax, iA^2B^2u)$$

\square

Using [18, Thm. 7] we can deduce the following corollary

Corollary 6.6. *One parameter families of double octic Calabi–Yau threefolds defined by arrangements No. No. 4, 13, 34, 72, 261 do not have a point of Maximal Unipotent Monodromy.*

The projective automorphism

$$(x, y, z, t, u) \mapsto (Bz, By, Bx, At, AB^3u)$$

transforms the equation

$$u^2 - xyz t(x + y)(y + z)(z + t)(Ax + Bt)$$

of the double octic no. 2 into

$$A^2 B^6 (u^2 - xyz t(x + y)(y + z)(z + t)(Bx + At))$$

the equation of the element corresponding to the parameter $(B : A)$ (multiplied by $A^2 B^6$), and so yields a horizontal transformation $(A : B) \mapsto (B : A)$ of the family. Consequently this family is a quadratic pull-back of another family of Calabi-Yau threefolds.

We found 54 examples of self transformations of one parameter families, in the table we collect only the transformation of the parameter.

Arr. No	$(A, B) \mapsto \dots$
2	(B, A)
4	$(A, A - B)$
5	$(A, 2A - B)$
10	$(A, -A - B)$
13	$(A, -A - B), (B, A), (B, -A - B), (A + B, -A), (A + B, -B)$
21	$(A, -A - B)$
34	$(A, -B), (B, A), (B, -A)$
36	$(A + B, -B)$
53	(B, A)
71	$(A, -A - B)$
72	$(A, A - B), (-2B + A, -B), (-2B + A, A - B)$
73	$(A, -A - B)$
96	$(A + B, -B)$
97	$(A + B, -B)$
98	$(A, A - B)$
99	$(A, -A - B)$
100	$(A, -A - B)$
144	$(A, -A - B)$
152	(B, A)
155	$(A, -A - B)$
198	$(A, -2A - B)$
200	$(A, -A - B), (B, A), (B, -A - B), (A + B, -A), (A + B, -B)$
242	$(A, -A - B), (A + 2B, -B), (A + 2B, -A - B)$

247	$(A + 2B, -B)$
248	$(A, -2A - B)$
249	$(A, -A - B)$
259	(B, A)
261	$(A, -B), (B, A), (B, -A)$
264	$(A, -B)$
267	$(A, A - B), (B, A), (B, B - A), (A - B, A), (A - B, -B)$
270	$(A + B, -2A - B)$
273	$(A + 2B, -B)$
274	(B, A)
276	(B, A)

Similarly, the projective automorphism

$$(x, y, z, t, u) \mapsto (A(y + z), -Ay, A(x + y), Bt, A^3Bu)$$

transforms the equation

$$u^2 - xyz t(x + y)(y + z)(Ax + By + Bz - At)(Ax + Ay + Bz - At)$$

of the double octic no. 2 into

$$A^6 B^2 (u^2 - (-\frac{B}{A})xyz t(x + y)(y + z)(Bx + Ay + Az - Bt)(Bx + By + Az - Bt))$$

the equation of the quadratic twist by $\sqrt{-\frac{B}{A}}$ of the element corresponding to the parameter $(B : A)$ (multiplied by $A^6 B^2$). We get only locally a transformation of the family overlying the map $(A : B) \mapsto (B : A)$.

We found 32 examples of twisted transformations of one parameter families, again in the table we collect only the transformation of the parameter.

Arr. No	$(A, B) \mapsto \dots$
4	$(B, A) (B, B - A) (A - B, A) (A - B, -B)$
16	$(A + B, -B)$
35	$(A, A - B)$
144	$(A + 2B, -B) (A + 2B, -A - B)$
153	$(A + B, -B)$
155	$(B, A) (B, -A - B) (A + B, -A) (A + B, -B)$
197	$(A, -A - B)$
244	(B, A)
246	$(A, -A - B)$

256	$(B, 4A)$
257	$(2A - B, -2B)$
264	$(B, -4A) (B, 4A)$
265	$(A, 4A - B)$
266	$(2A, -A - 2B) (4B, A) (4B, -A - 2B) (A + 2B, -B) (2A + 4B, -A)$
273	$(2A, -A - 2B) (4B, A) (4B, -A - 2B) (2A + 4B, -A)$
275	$(A - 3B, A + B) (A + 3B, B - A)$

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